The Lorentz Force

We can find empirically that a particle with mass m and electric charge q in an electric field $\mathbf{E}$ experiences a force $\mathbf{F}_E$ given by

$$\mathbf{F}_E = q\mathbf{E} \qquad \textbf{LF-1}$$

It is apparent from Equation LF-1 that, if q is a positive charge (e.g., a proton), $\mathbf{F}_E$ is parallel to, that is, in the direction of $\mathbf{E}$ and if q is a negative charge (e.g., an electron), $\mathbf{F}_E$ is antiparallel to, that is, opposite to the direction of $\mathbf{E}$ (see Figure LF-1). A positive charge moving parallel to $\mathbf{E}$ or a negative charge moving antiparallel to $\mathbf{E}$ is, in the absence of other forces of significance, accelerated according to Newton's second law:

$$\mathbf{F}_E = q\mathbf{E} = m\mathbf{a} \quad \Rightarrow \quad \mathbf{a} = \frac{q}{m}\mathbf{E} \qquad \textbf{LF-2}$$

Equation LF-2 is, of course, not relativistically correct. The relativistically correct force is given by

$$\mathbf{F}_E = q\mathbf{E} = \frac{d(\gamma m\mathbf{u})}{dt} = m\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\frac{d\mathbf{u}}{dt} = m\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\mathbf{a} \qquad \textbf{LF-3}$$

Classically, for example, suppose a proton initially moving at $v_0 = 10^3$ m/s enters a region of uniform electric field of magnitude $E = 500$ V/m antiparallel to the direction of $\mathbf{E}$ (see Figure LF-2*a*). How far does it travel before coming (instantaneously) to rest? From Equation LF-2 the acceleration slowing the proton is

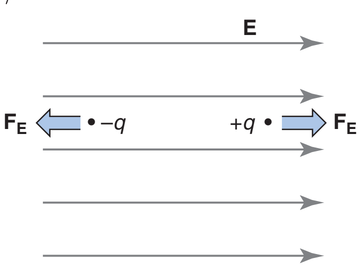
$$a = -\frac{q}{m}E = -\frac{(1.60 \times 10^{-19}\text{ C})(500\text{ V/m})}{1.67 \times 10^{-27}\text{ kg}} = -4.79 \times 10^{10}\text{ m/s}^2$$

The distance Δx traveled by the proton until it comes to rest with $v_f = 0$ is given by

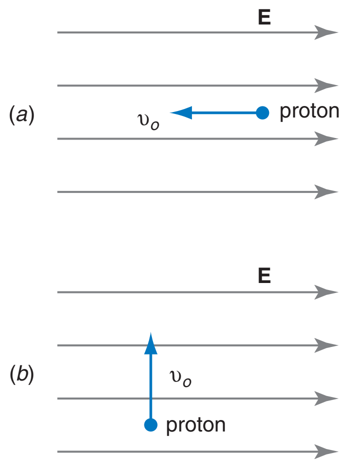
$$\Delta x = \frac{v_f^2 - v_0^2}{2a} = \frac{0 - (10^3\text{ m/s})^2}{2(-4.79 \times 10^{10}\text{ m/s}^2)}$$

$$\Delta x = 1.04 \times 10^{-5}\text{ m} = 1.04 \times 10^{-3}\text{ cm} \approx 0.01\text{ mm}$$

If the same proton is injected into the field perpendicular to $\mathbf{E}$ (or at some angle other than 0 or 180 degrees with respect to $\mathbf{E}$; see Figure LF-2*b*), it accelerates in the direction of the field following a parabolic path just like a projectile's motion in Earth's gravitational field.



LF-1 A positively charged particle in an electric field experiences a force in the direction of the field. The force on a negatively charged particle is opposite to the direction of the field.



LF-2 (*a*) A proton injected into an electric field opposite to the direction of the field experiences a force that slows it to rest (instantaneously), then is accelerated in the direction of the field. (*b*) A proton entering a uniform electric field perpendicular to the field lines receives acceleration in the field direction resulting in an increasing velocity component in that direction. The trajectory is a parabola, just as in the case of projectile motion in Earth's gravitational field.

We can also observe experimentally that an electric charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ experiences a force $\mathbf{F}_M$ given by

$$\mathbf{F}_M = q\mathbf{v} \times \mathbf{B} \qquad \textbf{LF-4}$$

The direction of $\mathbf{F}_M$ is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$. Being the result of a cross product, its direction is given by the right-hand rule for positive charges and opposite to that direction for negative charges. As an example, suppose an electron is moving with velocity $\mathbf{v} = 0.01c$ in the $+y$ direction perpendicular to a uniform magnetic field of 0.05 T in the $+x$ direction. Describe the electron's motion (see Figure LF-3*a*).

$$\begin{aligned} F_M &= evB \sin\theta \\ &= (1.60 \times 10^{-19}\,\mathrm{C})(0.01c)(0.05\,\mathrm{T}) \sin(\pi/2) \\ &= 2.40 \times 10^{-14}\,\mathrm{N} \perp \mathbf{v} \text{ and } \mathbf{B} \end{aligned}$$

The force $\mathbf{F}_M$ above is directed out of the plane of Figure LF-3*a*. The electron moves in a circle of radius r under the action of $\mathbf{F}_M$ as a centripetal force, with r given by

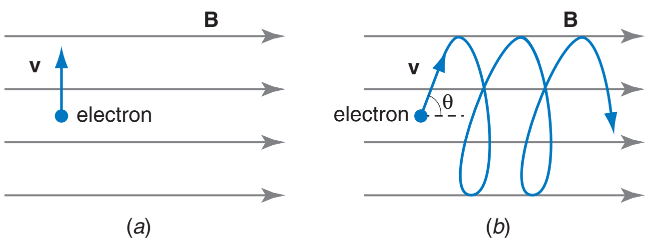
$$r = \frac{mv^2}{F_M} = \frac{(9.11 \times 10^{-31}\,\mathrm{kg})(0.01c)^2}{2.40 \times 10^{-14}\,\mathrm{N}} = 3.4 \times 10^{-4}\,\mathrm{m} = 0.34\,\mathrm{mm}$$

If $\mathbf{v}$ is not perpendicular to $\mathbf{B}$, then the component of $\mathbf{v}$ parallel to $\mathbf{B}$ $(=v \cos\theta)$ experiences no magnetic force and that part of the particle's motion at $v \cos\theta$ parallel to $\mathbf{B}$ does not change. However, the component of $\mathbf{v}$ perpendicular to $\mathbf{B}$ $(=v \sin\theta)$ moves in a circle of radius r as described above. The combined effect of the two motions is a helical trajectory about the $\mathbf{B}$ direction as illustrated in Figure LF-3*b*.

A charged particle moving in a region of space where both an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ exist experiences an electromagnetic force that is the sum of $\mathbf{F}_E$ and $\mathbf{F}_M$:

$$\mathbf{F} = \mathbf{F}_E + \mathbf{F}_M = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \textbf{LF-4}$$

The force $\mathbf{F}$ in Equation LF-4 is called the Lorentz force. Notice that $\mathbf{F}_E$ changes the magnitude of the velocity and $\mathbf{F}_M$ changes its direction. Equation LF-4 was used by H. A. Lorentz in measuring the charge-to-mass ratio e/m of the electron, and it has subsequently been employed in a great many practical applications.



LF-3 (*a*) The force acting on the electron moving perpendicular to the magnetic field is into the diagram. The resulting motion is a circle perpendicular to the plane of the figure. The magnitude of the velocity does not change. (*b*) If the electron (or any charged particle) moves at an angle other than 0° or 180° the velocity component parallel to $\mathbf{B}$ continues pointing in that direction while the component perpendicular to $\mathbf{B}$ rotates in a circle. The combined motion is a helix.